Math 279 Lecture 20 Notes

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1 Regularity Structures

1.1 Regularity structures and their relation to coherence

We have proved the reconstruction theorem, which says that if you have some local regularity, i.e. coherence, then you can construct a distribution which serves as a local approximation. Last time, we discussed the regularity structure, a bookkeeping device for discussing obstructions we deal with in a PDE: Namely, we have a discrete $A \subseteq \mathbb{R}$ with $0 \in A$ and $\min A > -\infty$ (the members of this set represent the homogeneity and hence regularity of various terms you have to deal with). We have a Banach space $T = \bigoplus_{r \in A} T_r$ with Banach spaces T_r with $\| \cdot \|_r$; we are mostly interested in when A is a finite set and T_r are Euclidean spaces. We always assume $T_0 = \mathbb{R}$; the dimension of these spaces will be the number of beasts of that type you have to deal with.

If we have a Taylor expansion at a point x we can re-expand to turn it into a Taylor expansion at a point y. We express this idea in this setting by a group G of linear, continuous transformations $\Gamma : T \to T$. Moreover, for $\tau \in T_r$, $\Gamma\tau - \tau \in \bigoplus_{s<r} T_s$. We also write

$$T_{<r} = \bigoplus_{s<r} T_s, \qquad T_{\leq r} = \bigoplus_{s \leq r} T_s.$$

Definition 1.1. Let $\mathcal{L}(T) = \mathcal{L}(T; \mathcal{D}')$ be the set of linear, continuous maps $L : T \to \mathcal{D}'$. We say $M = (\Pi, \Gamma)$ is a **model** for (A, T, G) if $\Pi : \mathbb{R}^d \to \mathcal{L}(T)$ and $\Gamma : \mathbb{R}^d \times \mathbb{R}^d \to G$ with the following properties (denoting $\Pi_x = \Pi(x), \Gamma_{x,y} = \Gamma(x, y)$):

1. $\Pi_x \tau = \pi_y \Gamma_{x,y} \tau$
2. $\Gamma_{x,y} \Gamma_{y,z} = \Gamma_{x,z}$.
3. If $\tau \in T_\alpha$, then

$$\sup_{\delta \in (0,1]} \sup_{\varphi \in \mathcal{D}_r} \sup_{x \in K} \frac{|(\Pi_x \tau)(\varphi_x^\delta)|}{\|\tau\|_\alpha \delta^\alpha} < \infty,$$

   where K is a compact set, and $\mathcal{D}_r = \{\varphi \in \mathcal{D} : \operatorname{supp} \varphi \subseteq B_1(0), \|\varphi\|_{C^r} \leq 1\}$, where r is the smallest integer that is more than $-\min A$.

4. $\|\Gamma_{x,y}\tau\|_\beta \lesssim |x-y|^{\alpha-\beta}\|\tau\|_\alpha$.

Π turns an abstract symbol into a distribution in a way that respects all this linear structure.

Definition 1.2. Next, we define $\mathcal{C}_M^\gamma$ to be the set of functions $f : \mathbb{R}^d \to T_{<\gamma}$ such that

$$\|f(x) - \Gamma_{x,y}f(y)\|_\alpha \lesssim |x-y|^{\gamma-\alpha}$$

for $\alpha < \gamma$.

Hence, we may define the norm

$$[f]_\alpha = \sup_{\alpha<\gamma} \sup_{x\neq y\in K} \frac{\|f(x) - \Gamma_{x,y}f(y)\|_\alpha}{|x-y|^{\gamma-\alpha}}.$$

Theorem 1.1. *For every γ, there exists an operator $\mathcal{R}_\gamma : \mathcal{C}_M^\gamma \to \mathcal{D}'$ which is linear and continuous and which satisfies*

$$|(\mathcal{R}_\gamma f - \Pi_x f(x))(\varphi_x^\delta)| \lesssim \begin{cases} \delta^\gamma & \gamma \neq 0 \\ |\log\delta| & \gamma = 0, \end{cases}$$

uniformly for $\psi \in \mathcal{D}_r, \delta \in (0,1], x \in K$.

Proof. To simplify the notation, we write $f_x = f(x)$ and $\Pi_x = \Pi(x)$. Now define $F_x = \Pi_x(f_x)$. We can achieve the desired result if we can show that the germ $(F_x : x \in \mathbb{R}^d)$ is γ-coherent. Indeed,

$$\begin{aligned} |(F_x - F_y)(\varphi_y^\delta)| &= |(\Pi_x f_x - \Pi_y f_y)(\varphi_y^\delta)| \\ &= |(\Pi_x f_x - \Pi_x \Gamma_{x,y} f_y)(\varphi_y^\delta)| \\ &= |\Pi_x(f_x - \Gamma_{x,y}f_y)(\varphi_y^\delta)| \end{aligned}$$

Recall that $f : \mathbb{R}^d \to \bigoplus_{\alpha<\gamma} T_\alpha$, so $\|f_x - \Gamma_{x,y}f_y\| \lesssim |x-y|^{\gamma-\alpha}$. Here, $P_\alpha\tau$ means the α-component of τ, i.e. $\tau = \sum_{\alpha\in A}(P_\alpha\tau)$.

$$\begin{aligned} &= \left|\prod_x P_\alpha(f_x - \Gamma_{x,y}f_y)(\varphi_y^\delta)\right| \\ &\leq \sum_{\alpha<\gamma} \left|\prod_x \sum_{\alpha<\gamma} P_\alpha(f_x - \Gamma_{x,y}f_y)(\varphi_y^\delta)\right| \\ &\lesssim \sum_{\alpha<\gamma} \delta^\alpha \|f_x - \Gamma_{x,y}f_y\|_\alpha \end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{\alpha<\gamma} \delta^{\alpha}|x-y|^{\gamma-\alpha} \\
&= \delta^{-r} \sum_{\alpha<\gamma} \delta^{\alpha+r}|x-y|^{\gamma-\alpha} \\
&= \delta^{-r} \sum_{\alpha<\gamma} (\delta+|x-y|)^{\gamma+r} \\
&\lesssim \delta^{-r}(\delta+|x-y|)^{\gamma+r},
\end{aligned}
$$

which is exactly the definition of coherence.[1] □

1.2 An example: Taylor series

Example 1.1. Assume $A = \{0, 1, 2, \dots\}$, and let $T = \mathbb{R}[X_1, \dots, X_d]$ be the space of polynomials of variables $X_1, \dots, X_d$ with real coefficients. Again, we use $X^k = X_1^{k_1} \cdots X_d^{k_d}$, where $k = (k_1, \dots, k_d) \in \mathbb{N}^d$ and $|k| = k_1 + \cdots + k_d$. Now T_r is the set of homogeneous polynomials of degree r, $\operatorname{span}(\{X^k : |k| = r\}) = \langle X^k : |k| = r \rangle$, and $T_{\leq r}$ is the set of polynomials of degree $\leq r$. (By $S = \langle \tau^1, \dots, \tau^e ll \rangle$, we mean $S = \operatorname{span}(\tau^1, \dots \tau^\ell)$ and $\tau^1, \dots, \tau^\ell$ are linearly independent.)

Next, $G = \{\Gamma_h : h \in \mathbb{R}^d\}$. Formally,

$$\Gamma_h(P(X)) = P(X + h\mathbf{1}).$$

For example, $(X + h\mathbf{1})^k = \prod_{i=1}^d (X_i + h\mathbf{1})^{r_i}$, using the convention that $X_i\mathbf{1} = \mathbf{1}X_i = X_i$. If $\deg P = r$, then $\deg(\Gamma_h P - P) < r$.

We now define a model for this:

$$\Pi_a(P(X))(x) = P(x - a).$$

Observe that

$$
\begin{aligned}
\langle \Pi_a(X^k), \varphi_a^\delta \rangle &= \int \Pi_a(X^k)(x)\varphi_a^\delta(x) \\
&= \int (x-a)^k \varphi_a^\delta \, dx
\end{aligned}
$$

Use a change of variables.

$$= \left(\int P(x)\varphi(x)\, dx \right) \delta^{|k|}$$

Next, we discuss $\mathcal{C}_M^\gamma$ with $\gamma = n + \gamma_0$, where $n \in \mathbb{N}$ and $\gamma_0 \in (0, 1)$. Let $f \in \mathcal{C}_M^\gamma$. Then $f(x) = \sum_k c_k(x) X^k \in \bigoplus_{r<\gamma} T_r$ is a polynomial of degree n. We claim that we must have that $c_k = \frac{1}{k!}\partial^k c_0$ with $c_0 \in \mathcal{C}^\gamma$ and c_k must be Hölder of exponent γ_0 when $|k| = n$.

[1] Professor Rezakhanlou described this as a "one-line proof."

This is the same flavor as in the Whitney extension theorem. First, we have the Tietze extension theorem. If we have a closed subset of a decent topological space, we can extend a continuous function on the closed subset to the whole set without increasing the norm of it. The Whitney extension theorem achieves this with derivatives by assigning polynomials to each point and showing that they must be related via Taylor expansion.